Indian Statistical Institute
Mid-Sem Examination
Differential Topology- MMath II

Max Marks : 40 06.10.10

Answer all questions. You may use the results proved in class, to justify your claims, after correctly stating them. Any other claim must be accompanied by a proof.

(1) Decide whether the following statements are TRUE or FALSE. Justify.

(a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be given by $f(x, y) = (x + e^y, e^x)$ and $Z = \{(t, t^2)/t \in \mathbb{R}\}$, then $f^{-1}(Z)$ is a submanifold of $\mathbb{R}^2$.

(b) The function $f : S^1 \longrightarrow \mathbb{R}$ given by $f(x, y) = x + y$ has four critical points.

(c) If $f, g : M \longrightarrow N$ are smooth maps between smooth manifolds, then

$$A = \{y \in N : y \text{ is a regular value of both } f \text{ and } g\}$$

is dense in N.

(d) The determinant function $det : M(3, \mathbb{R}) \longrightarrow \mathbb{R}$ is Morse.

(e) If $f : M \longrightarrow N$ is a smooth map between manifolds and

$$I_f = \{x \in M : f \text{ is an immersion at } x\},$$

then I_f is open in M [5×2=10]

(2) Show that the set M of 2×2 real matrices of rank one is a submanifold of $M(2, \mathbb{R})$. Dtermine the dimension of M and describe the tangent space to M at the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. [10]

(3) Let M be a connected smooth manifold and $p, q \in M$ be two distinct points. Show that there exists a diffeomorphism $\varphi : M \longrightarrow M$ with $\varphi(p) = q$. [8]

(4) Show that every non constant smooth (real valued) function on a compact manifold has at least two critical points. Further show that the non degenerate critical points of a smooth function are isolated. [6]

(5) Let $U \subseteq \mathbb{R}^n$ be open and $f : U \longrightarrow \mathbb{R}$ smooth. For any point $x \in U$, let $H(x)$ denote the Hessian of f at x. Show that f is Morse if and only if

$$det\ H^2 + \Sigma_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2 > 0$$

on U. [4]